Heat & Mass Flow Processes

Week_04

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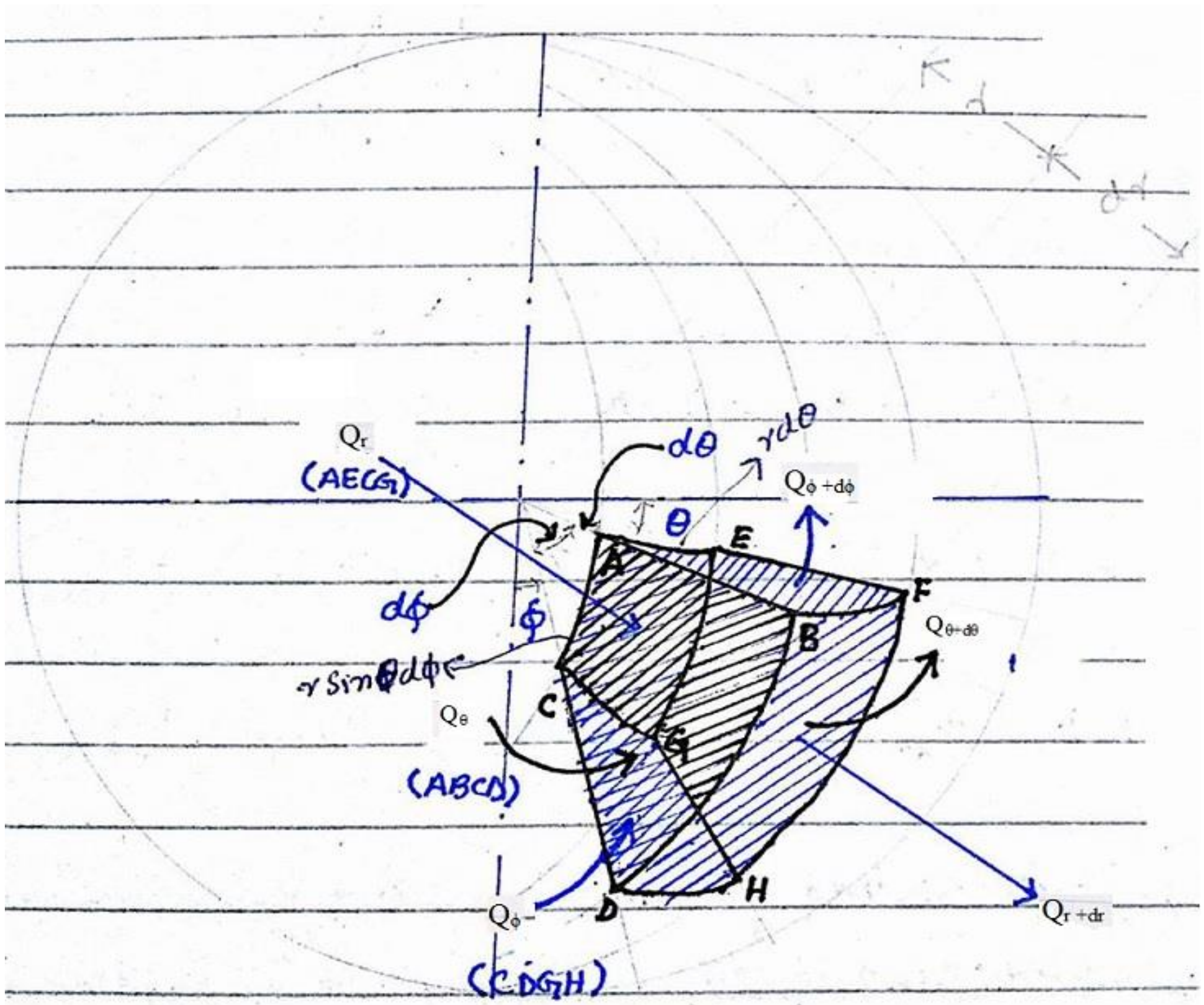
Heat Diffusion Equation in Spherical Coordinate System

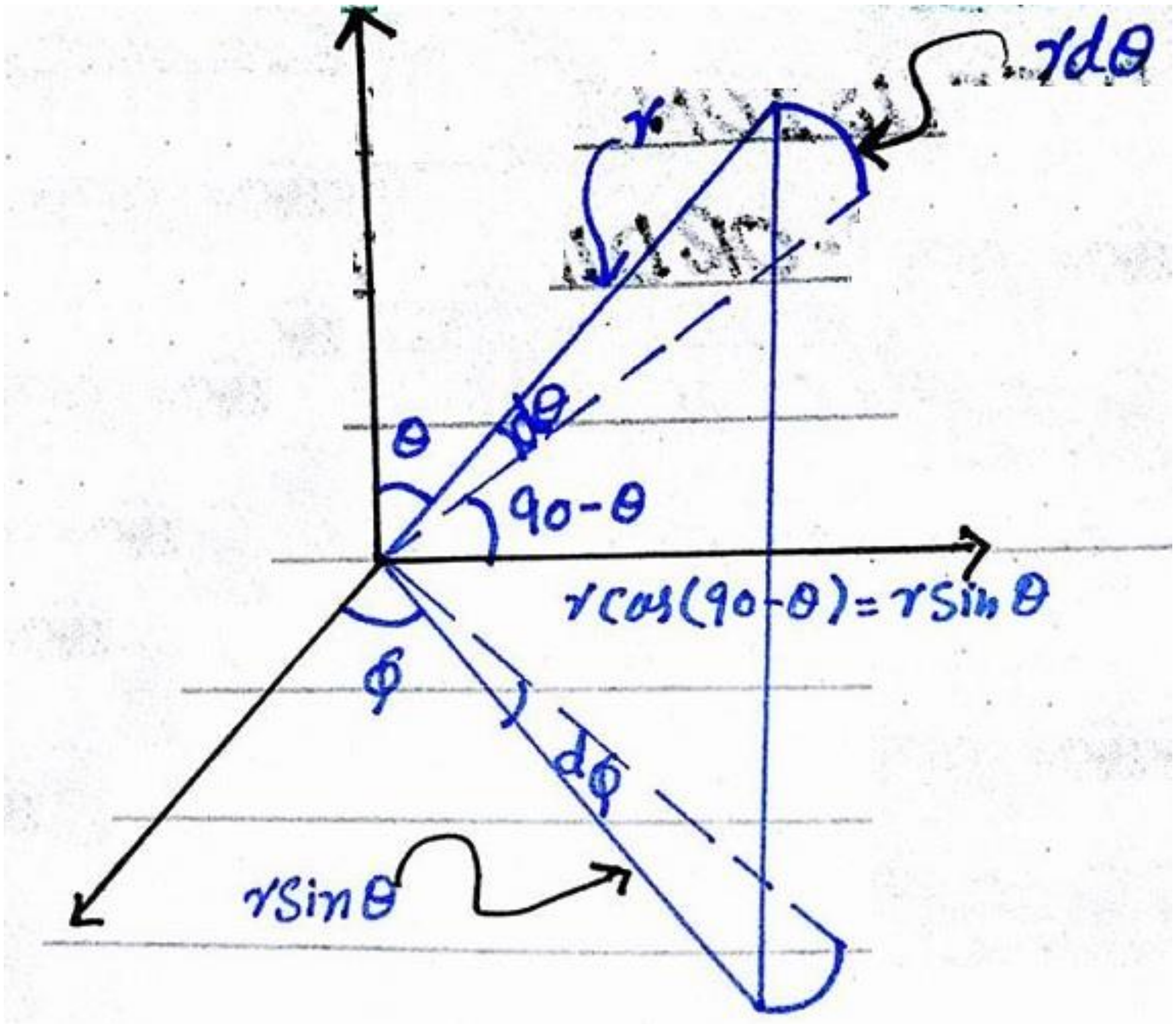
- Let us consider a small element of a sphere as shown in Fig. 1 in which heat flows in all three directions r, θ, and ϕ as shown in Fig. below;

Coordinates:	r	:	θ	:	ϕ
Heat Transfer:	Q_r	:	Q_θ	:	Q_ϕ
Heat Flux (q=Q/A)	q_r	:	q_θ	:	q_ϕ
Thickness:	dr	:	$r d\theta$	:	$r\sin\theta\, d\phi$

Heat Diffusion Equation in Spherical Coordinate System



Heat Diffusion Equation in Spherical Coordinate System



Heat Diffusion Equation in Spherical Coordinate System

- Energy balance for the small element is obtained from the First law of Thermodynamics.

(Net heat conducted into the element in dr, rdθ, rsinθdφ per unit time I) + (Internal heat generated per unit time II)

= (Increase in internal energy per unit time III) + (work done by element per unit time IV)

- The IV term is small because the work done by the element due to change in temperature in neglected.

Qin– Qout + Qgen. = Rate of change in Internal Energy

Heat Diffusion Equation in Spherical Coordinate System

$$(Q_r + Q_\theta + Q_\phi) - (Q_{r+dr} + Q_{\theta+d\theta} + Q_{\phi+d\phi}) + (Qgen.) = mc\frac{\partial T}{\partial \tau}$$

$$(Q_r + Q_\theta + Q_\phi) - (Q_{r+dr} + Q_{\theta+d\theta} + Q_{\phi+d\phi}) + (Qgen.) = \rho v c\frac{\partial T}{\partial \tau} \rightarrow (1)$$

Let

- Q_r be the heat flux in "r" direction at face AECG and Q_{r+dr} be the heat flux at r+dr direction at face BDFH.
- Q_θ be the heat flux in "θ" direction at face ABCD and $q_{\theta+d\theta}$ be the heat flux at θ+dθ direction at face EFGH.
- Q_ϕ be the heat flux in "ϕ" direction at face DHCG and $Q_{\phi+d\phi}$ be the heat flux at ϕ+dϕ direction at face AEBF

Heat Diffusion Equation in Spherical Coordinate System

Now the rate of heat transfer in 'r', 'θ', and 'ϕ' direction is given by:

$$Q_r = -k.r^2 d\theta \sin\theta d\phi \frac{\partial T}{\partial r}; Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)dr + ...$$

$$Q_\theta = -k.rdr\sin\theta d\phi \frac{\partial T}{rd\theta}; Q_{\theta+d\theta} = Q_\theta + \frac{\partial}{\partial\theta}(Q_\theta)d\theta + ...$$

$$Q_\phi = -k.rd\theta dr \frac{dT}{r\sin\theta d\phi}; Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{\partial\phi}(Q_\phi)d\phi + ...$$

Now by putting all values in Equation (1) we have.

$$-\frac{\partial}{\partial r}(Q_r)dr - \frac{\partial}{\partial\theta}(Q_\theta)d\theta - \frac{\partial}{\partial\phi}(Q_\phi)d\phi + q'''.V = \rho Vc\frac{\partial T}{\partial\tau}$$

$$-\frac{\partial}{\partial r}(-k.r^2 d\theta\sin\theta d\phi\frac{\partial T}{\partial r})dr - \frac{\partial}{\partial\theta}(-k.rdr\sin\theta d\phi\frac{\partial T}{r\partial\theta})d\theta$$

$$-\frac{\partial}{\partial\phi}(-k.rd\theta dr\frac{\partial T}{r\sin\theta\partial\phi})d\phi + q'''.V = \rho Vc\frac{\partial T}{\partial\tau}$$

$$k.drd\theta d\phi\sin\theta\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right] + k.drd\theta d\phi\frac{\partial}{\partial\theta}\left[\sin\theta d\phi\frac{\partial T}{\partial\theta}\right]$$

$$+\frac{k.drd\theta d\phi}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial T}{\partial\phi})\right] + q'''.V = \rho Vc\frac{\partial T}{\partial\tau}$$

Heat Diffusion Equation in Spherical Coordinate System

Since volume=v=dr.rdθ.rsinθdϕ, hence by dividing both sides of the above equation by "k.V", it is obtained that;

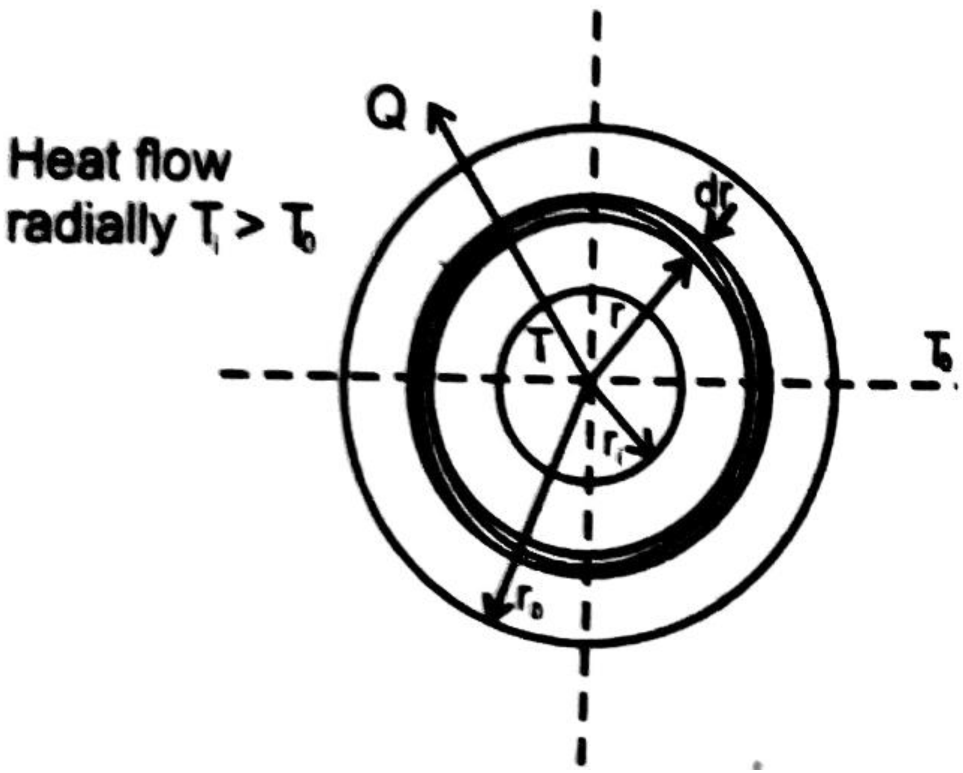
$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}[r^2\frac{\partial T}{\partial r}]+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial T}{\partial\theta}]$$

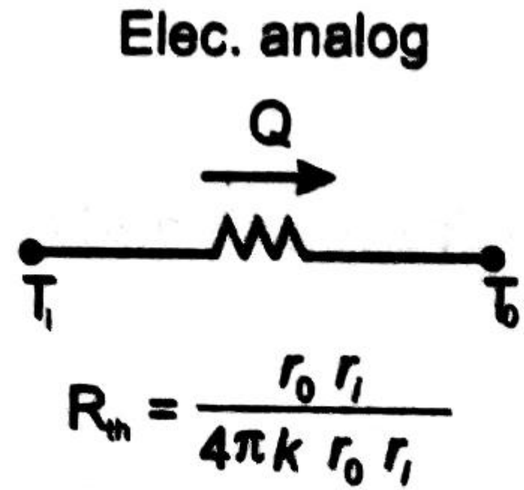
$$+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\phi}(\frac{\partial T}{\partial\phi})+\frac{q'''}{k}=\frac{1}{\alpha}\cdot\frac{dT}{d\tau}\rightarrow(2)$$

This equation is called unsteady state equation for spherical coordinate system.

Heat Conduction Through a Sphere without Internal Generation

- ✰ Consider a hollow sphere, whose inside and outside radii are r_i and r_0 respectively.
- ✰ Inner and outer surfaces are maintained at temp T_i and T_0 with uniform thermal conductivity k.





Heat Conduction Through a Sphere without Internal Generation

Assume no heat generation, steady state, temperature variation only in radial direction.
From general heat conduction equation;

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right]$$

$$+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial T}{\partial\phi}\right)+\frac{q'''}{k}=\frac{1}{\alpha}\cdot\frac{dT}{d\tau}\rightarrow(3)$$

Hence, equation (3) can be written as in terms of radius "r";

$$\boxed{\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{q'''}{k}=\frac{1}{\alpha}\cdot\frac{\partial T}{\partial\tau}}\rightarrow(4)$$

Heat Conduction Through a Sphere without Internal Generation

Now heat conduction equation becomes.

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial T}{\partial r} = o$$

Multiply by r^2 throughout, we get

$$r^2 \frac{d^2T}{dr^2} + 2r \cdot \frac{dT}{dr} = 0$$

(or) $$\frac{d}{dr}\left(r^2 \cdot \frac{dT}{dr}\right) = 0$$

Hence $$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

Heat Conduction Through a Sphere without Internal Generation

Integrating the above equation,

$$\therefore \qquad r^2 \cdot \frac{dT}{dr} = C_1$$

$$\frac{dT}{dr} = \frac{C_1}{r^2}$$

Integrating again

$$\boxed{T = -\frac{C_1}{r} + C_2} \quad \rightarrow (5)$$

To find C_1 and C_2 apply boundary conditions

$$\therefore \qquad T = T_i \quad \text{at} \quad r = r_i \qquad \therefore \qquad T_i = -\frac{C_1}{r_i} + C_2 \quad \rightarrow (6)$$

$$\therefore \qquad T = T_0 \quad \text{at} \quad r = r_0 \qquad \therefore \qquad T_0 = -\frac{C_1}{r_0} + C_2 \quad \rightarrow (7)$$

Heat Conduction Through a Sphere without Internal Generation

Solving for C_1 and C_2

$$\therefore \quad C_1 = \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)} \qquad \therefore \quad C_2 = T_i + \frac{1}{r_i}\frac{(T_i - T_0)}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)}$$

$$\therefore \quad T = \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right) r} + \left(T_i + \frac{1}{r_i}\frac{(T_i - T_0)}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)}\right)$$

$$\boxed{T = T_i - \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)}\left(\frac{1}{r} - \frac{1}{r_i}\right)} \quad \rightarrow (8)$$

T is called temperature distribution for hollow sphere.

Heat Conduction Through a Sphere without Internal Generation

To find heat flow,

$$Q = -kA \cdot \frac{dT}{dr} = -k \cdot 4\pi r^2 \cdot \frac{dT}{dr}$$

$$Q \cdot \frac{dr}{r^2} = -k4\pi \cdot dT$$

Integrating the above equation

$$Q \int_{r_i}^{r_0} \frac{dr}{r^2} = -4\pi k \int_{T_i}^{T_0} dT$$

$$-Q\left[\frac{1}{r_0} - \frac{1}{r_i}\right] = -4\pi k\,(T_0 - T_i)$$

$$-Q\left[\frac{r_i - r_0}{r_0 r_i}\right] = -4\pi k\,(T_0 - T_i)$$

$$\boxed{Q = \frac{4\pi k r_0 r_i (T_i - T_0)}{r_0 - r_i}} \qquad \rightarrow (9)$$

Heat Conduction Through a Sphere without Internal Generation

Thermal resistance for a spheral system is

$$\boxed{R_{\text{th. sp.}} = \frac{r_0 - r_i}{4\pi k r_0 r_i}} \quad \rightarrow (10)$$

Heat Conduction Through a Sphere with Internal Generation

- ☆ Let us consider a sphere of radius R and uniform thermal conductivity k.
- ☆ Consider an elemental ring of radius r and thickness dr. Heat generated in dr is Q_g.
- ☆ Let us electric current is pass through the sphere, causes uniform that generation within the sphere.
- ☆ Let the surface temperature of the sphere be T_w. Heat generated in the sphere be $q'''W/m^3$.

From general heat conduction equation;

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right]$$

$$+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial T}{\partial\phi}\right)+\frac{q'''}{k}=\frac{1}{\alpha}\cdot\frac{dT}{d\tau}\rightarrow\quad(11)$$

Heat Conduction Through a Sphere with Internal Generation

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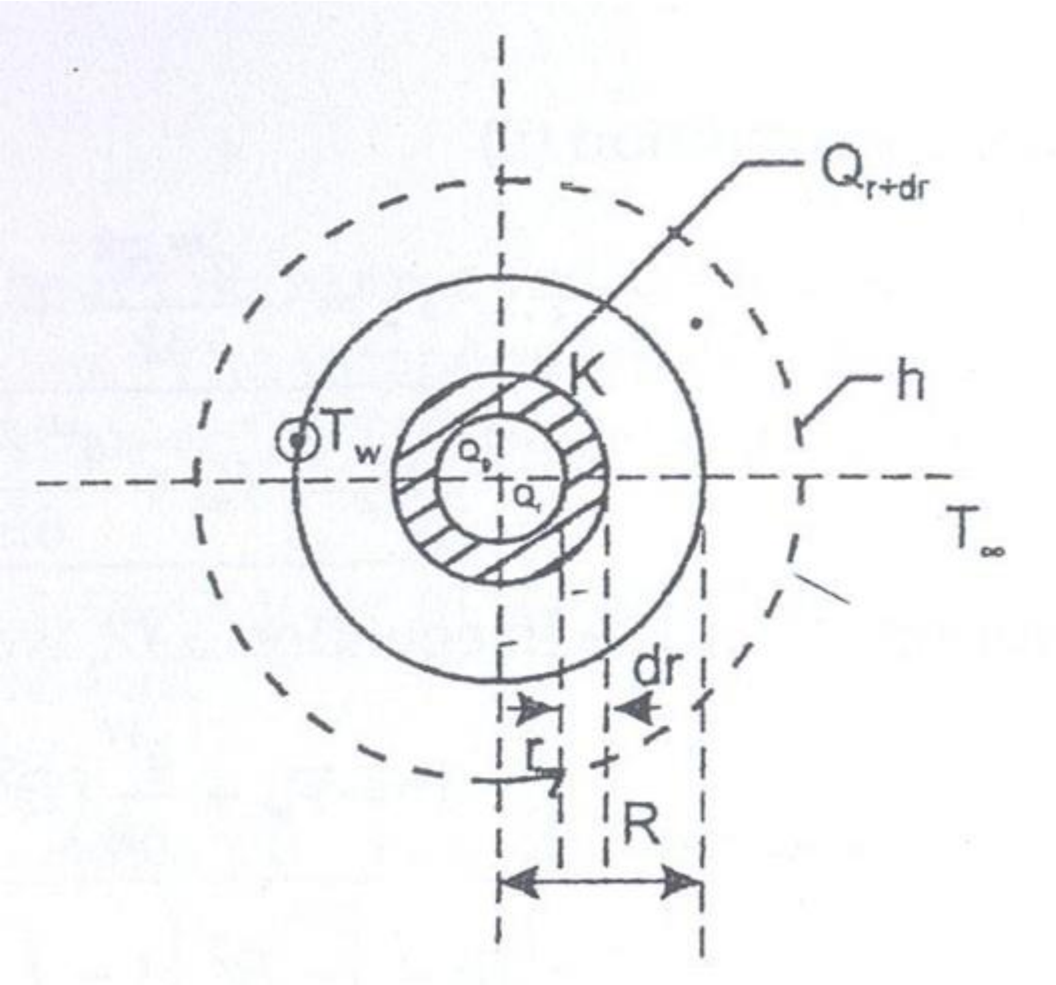


Heat Conduction Through a Sphere with Internal Generation

Since there is steady state conduction in radial direction only with internal heat generation, all the other terms will become equal to zero. Hence equation (14) will become;

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}[r^2\frac{\partial T}{\partial r}]+\frac{q'''}{k}=0 \Rightarrow \frac{\partial}{\partial r}[r^2\frac{\partial T}{\partial r}]+\frac{q'''r^2}{k}=0 \Rightarrow \frac{d}{dr}[r^2\frac{dT}{dr}]+\frac{q'''r^2}{k}=0 \rightarrow (12)$$

Integrating the above equation

$$r^2\cdot\frac{dT}{dr}=-\frac{q'''r^3}{3k}+C_1$$

again integrating we get,

$$\int\frac{dT}{dr}=-\frac{q'''}{3k}\int\frac{r^3}{r^2}+\int\frac{C_1}{r^2}$$

$$\boxed{T=-\frac{q'''r^2}{6k}-\frac{C_1}{r}+C_2} \rightarrow (13)$$

Heat Conduction Through a Sphere with Internal Generation

To find C_1 and C_2

Apply Boundary Cor

i. $\frac{dT}{dr} = 0$ at $r = 0$

ii. $T = T_w$ at $r = R$

By first boundary condition

$$\boxed{C_1 = 0}$$

Now equation 5 becomes,

$$T = -\frac{q''' r^2}{6k} + C_2$$

Heat Conduction Through a Sphere with Internal Generation

apply boundary condition (ii)

$$\therefore \quad T_2 = -\frac{q''' R^2}{6k} + C_2$$

$$\therefore \quad \boxed{C_2 = T_w + \frac{q''' R^2}{6k}}$$

Substituting C_1 and C_2 in equation (13), we get

$$T = T_w + \frac{q'''}{6k}(R^2 - r^2)$$

$$\boxed{T = T_w + \frac{q'''}{6k} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]} \rightarrow (14)$$

At centre temperature $T = T_c$ and $r = 0$

$$\therefore \quad \boxed{T_c = T_w + \frac{q''' R^2}{6k}}$$

Heat Conduction Through a Sphere with Internal Generation

centre temperature T_c is also called Maximum Temperature

$$\therefore \quad \boxed{T_{\max} = T_w + \frac{q''' R^2}{6k}} \rightarrow (15)$$

To find wall or surface temperature

Heat generated in the sphere

$$(Q) = V \times q'''$$

V : Volume

$$Q = \frac{4}{3}\pi R^3 \cdot q''' \rightarrow (16)$$

Heat transfer due to convection on the surface of the sphere

$$Q = h\,A(T_w - T_\infty)$$

$$\boxed{Q = h\,4\pi R^2 (T_w - T_\infty)} \rightarrow (17)$$

Heat Conduction Through a Sphere with Internal Generation

Equating equation (16) and (17);

$$\therefore \quad \frac{4}{3}\pi R^3 \cdot q''' = 4\pi R^2 h(T_w - T_\infty)$$

$$\therefore \quad \boxed{T_w = T_\infty + \frac{q''' R}{3h}} \rightarrow (18)$$

where,

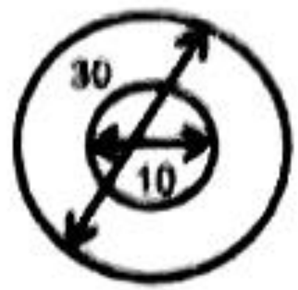
T_w : surface temperature of the sphere.

Conduction Heat Transfer-Class Problems

Example 4.1:

A hollow sphere 10 cm I.D and 30 Cm O.D of a material having thermal conductivity 50 W/mK is used as a container for a liquid chemical mixture. Its inner and outer surface temperatures are $300^0\ C$ and $100^0\ C$ respectively. Determine the heat flow rate through the sphere. Also estimate the temperature at a point quarter of the way between inner and outer surface.

$$R_{th} = \frac{(r_0 - r_i)}{4\pi r_0 r_i k}$$



Conduction Heat Transfer-Class Problems

Solution

$$Q = \frac{4\pi r_o r_i k(T_i - T_o)}{(r_o - r_i)} = \frac{(12.56)(0.15)(0.05)(50)(300 - 100)}{(0.15 - 0.05)}$$

$$\boxed{Q = 9.42kW}$$

The value of r at one-fourth way of inner and outer surface is

$\therefore \quad r = 7.5cm \quad \therefore$ Temperature at $r = 7.5cm$ i.e., $5 + \frac{1}{4}(15-5) = 7.5cm$

$$T = \frac{r_o}{r}\frac{(r - r_i)}{(r_o - r_i)}(T_o - T_i) + T_i$$

i.e., $$\frac{4\pi k r_i r_o(T_i - T_o)}{r_o - r_i} = \frac{4\pi k r r_i(T - T_i)}{r - r_i}$$

$$T = \frac{r_o}{r}\frac{(r - r_i)}{(r_o - r_i)}(T_o - T_i) + T_i$$

$$T = \frac{0.15}{0.075}\frac{(0.075 - 0.05)}{(0.15 - 0.05)}(100 - 300) + 300 = 200^\circ C$$

Conduction Heat Transfer-Class Problems

Example 4.2:

The average heat produced by oranges ripening is estimated to be 300 W/m^2. Taking the average size of an orange to be 8 cm and assuming it to be a sphere, with $k = 0.15$ W/mK, calculate the temperature at the centre of the orange.

Given:

$$Q/A = 300\ W/m^2$$
$$D = 8\ cm = 0.08\ m$$
$$R = 0.04\ m$$
$$k = 0.15\ W/mK$$

Find:

T_{max}

Conduction Heat Transfer-Class Problems

Solution

We know that,

$$\frac{Q}{A} = q''' \times \frac{Volume}{Area}$$

$$= \frac{q''' \times 4/3\pi R^3}{4\pi R^2}$$

$$\frac{Q}{A} = \frac{q'''}{3} R$$

$$\therefore \quad q''' = \frac{3Q}{AR}$$

$$= \frac{3 \times 300}{0.04}$$

Heat generated $\boxed{q''' = 2.25 \times 10^4 \; W/m^3}$

Conduction Heat Transfer-Class Problems

$\therefore$ Temperature at the center of the sphere

$$T_{\max} = T_\infty + \frac{q''' R^2}{6k}$$

$$= 10 + \frac{2.25 \times 10^4 (0.04)^2}{6 \times 0.15}$$

$$\boxed{T_{\max} = 50°C}$$

Links for Video Lectures

Theory Class:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20THEORY-20210502_101529-Meeting%20Recording.mp4?web=1

Lab Session_04:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20LAB-20210502_123847-Meeting%20Recording.mp4?web=1


Thank You!