Heat & Mass Flow Processes

Week_03

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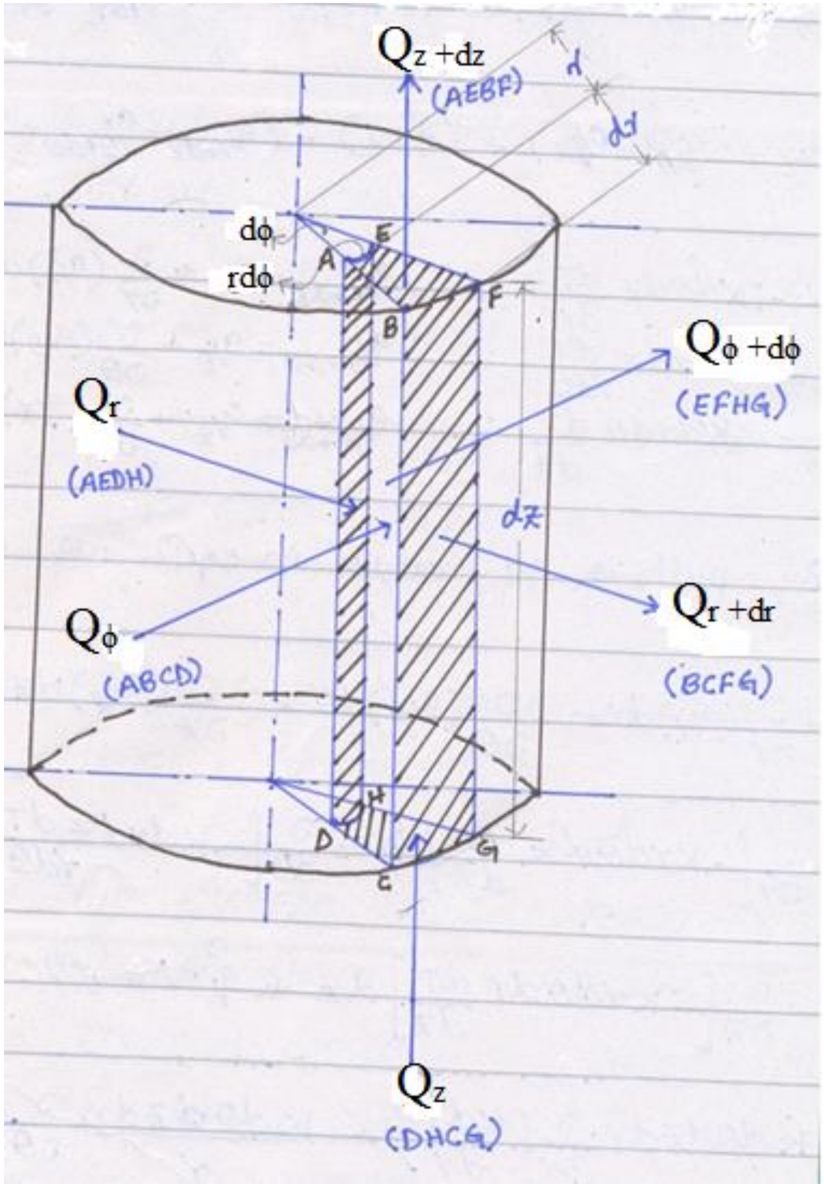
Heat diffusion Equation In Cylindrical Coordinate system

Let us consider a small element cut out of a pipe as shown in Fig 2.2. as given below in which heat enters from three faces and leaves the object from the other three faces in r, ϕ and z directions respectively.

Coordinates:	r	:	ϕ	:	Z
Heat Transfer:	Qr	:	$Q\phi$	:	Qz
Heat Flux (q=Q/A)	qr	:	$q\ \phi$	:	qz
Thickness:	dr	:	$rd\phi$	:	dz

Heat diffusion Equation In Cylindrical Coordinate system



- Energy balance for the small element is obtained from the First law of Thermodynamics.

(Net heat conducted into the element in dr, rdϕ, dz per unit time I) + (Internal heat generated per unit time II)

= (Increase in internal energy per unit time III) + (work done by element per unit time IV)

- The IV term is small because the work done by the element due to change in temperature in neglected.

$$Q_{in} – Q_{out} + Q_{gen.} = \text{Rate of change in Internal Energy}$$

Heat diffusion Equation In Cylindrical Coordinate system

$$(Q_r + Q_\phi + Q_z) - (Q_{r+dr} + Q_{\phi+d\phi} + Q_{z+dz}) + (Qgen.) = mc\frac{\partial T}{\partial \tau}$$

$$(Q_r + Q_\phi + Q_z) - (Q_{r+dr} + Q_{\phi+d\phi} + Q_{z+dz}) + (Qgen.) = \rho vc\frac{\partial T}{\partial \tau} \rightarrow (1)$$

Let

- Q_r be the heat flux in "r" direction at face AEDH and Q_{r+dr} be the heat flux at r+dr direction at face BCFG.
- Q_ϕ be the heat flux in "ϕ" direction at face ABCD and $q_{\phi+d\phi}$ be the heat flux at ϕ+dϕ direction at face EFHG.
- Q_z be the heat flux in "z" direction at face DHCG and Q_{z+dz} be the heat flux at z+dz direction at face AEBF

Now the rate of heat transfer in 'r', 'ϕ', and 'z' direction is given by:

Heat diffusion Equation In Cylindrical Coordinate system

$$\left.\begin{aligned} Q_r &= q_r r d_\phi d_z = -k\ r d_\phi d_z \frac{\partial T}{\partial r} \\ Q_\phi &= q_\phi d_r d_z = -k\ d_r d_z \frac{\partial T}{r\partial \phi} \\ Q_z &= q_z r d_r d_\phi = -k\ r d_r d_\phi \frac{\partial T}{\partial z} \end{aligned}\right\} \rightarrow (2)$$

Similarly, the rate of heat transfer in ‘r+dr’, ‘ϕ+dϕ’, and ‘z+dz’ direction is given by:

$$\left.\begin{aligned} Q_{r+dr} &= Q_r + \frac{\partial Q}{\partial r} d_r \ldots\ldots \\ Q_{\phi+d\phi} &= Q_\phi + \frac{\partial Q}{\partial \phi} d_\phi \ldots\ldots \\ Q_{z+dz} &= Q_z + \frac{\partial Q}{\partial z} d_z \ldots\ldots \end{aligned}\right\} \rightarrow (3)$$

Heat diffusion Equation In Cylindrical Coordinate system

Now by putting Equation (3) in Equation (1) we have.

$$-\frac{\partial}{\partial r}(Q_r)dr-\frac{\partial}{\partial \phi}(Q_\phi)d\phi-\frac{\partial}{\partial z}(Q_z)dz+q'''.v=\rho vc\frac{\partial T}{\partial \tau}$$

$$-\frac{\partial}{\partial r}(-krd_\phi d_z\frac{\partial T}{\partial r})dr-\frac{\partial}{r\partial \phi}(-kd_r d_z\frac{\partial T}{\partial \phi})d\phi-\frac{\partial}{\partial z}(-krd_r d_\phi\frac{\partial T}{\partial z})dz+q'''.v=\rho vc\frac{\partial T}{\partial \tau}$$

$$\frac{\partial}{\partial r}(krd_\phi d_z\frac{\partial T}{\partial r})dr+\frac{\partial}{\partial \phi}(kd_r d_z\frac{\partial T}{r\partial \phi})d\phi+\frac{\partial}{\partial z}(krd_r d_\phi\frac{\partial T}{\partial z}{}_z)dz+q'''.v=\rho vc\frac{\partial T}{\partial \tau}$$

$$kd_r d_\phi d_z\frac{\partial}{\partial r}(\frac{r\partial T}{\partial r})+\frac{kd_r d_\phi d_z}{r}\frac{\partial}{\partial \phi}(\frac{\partial T}{\partial \phi})+krd_r d_\phi d_z\frac{\partial}{\partial z}(\frac{\partial T}{\partial z})+q'''.v=\rho vc\frac{\partial T}{\partial \tau}$$

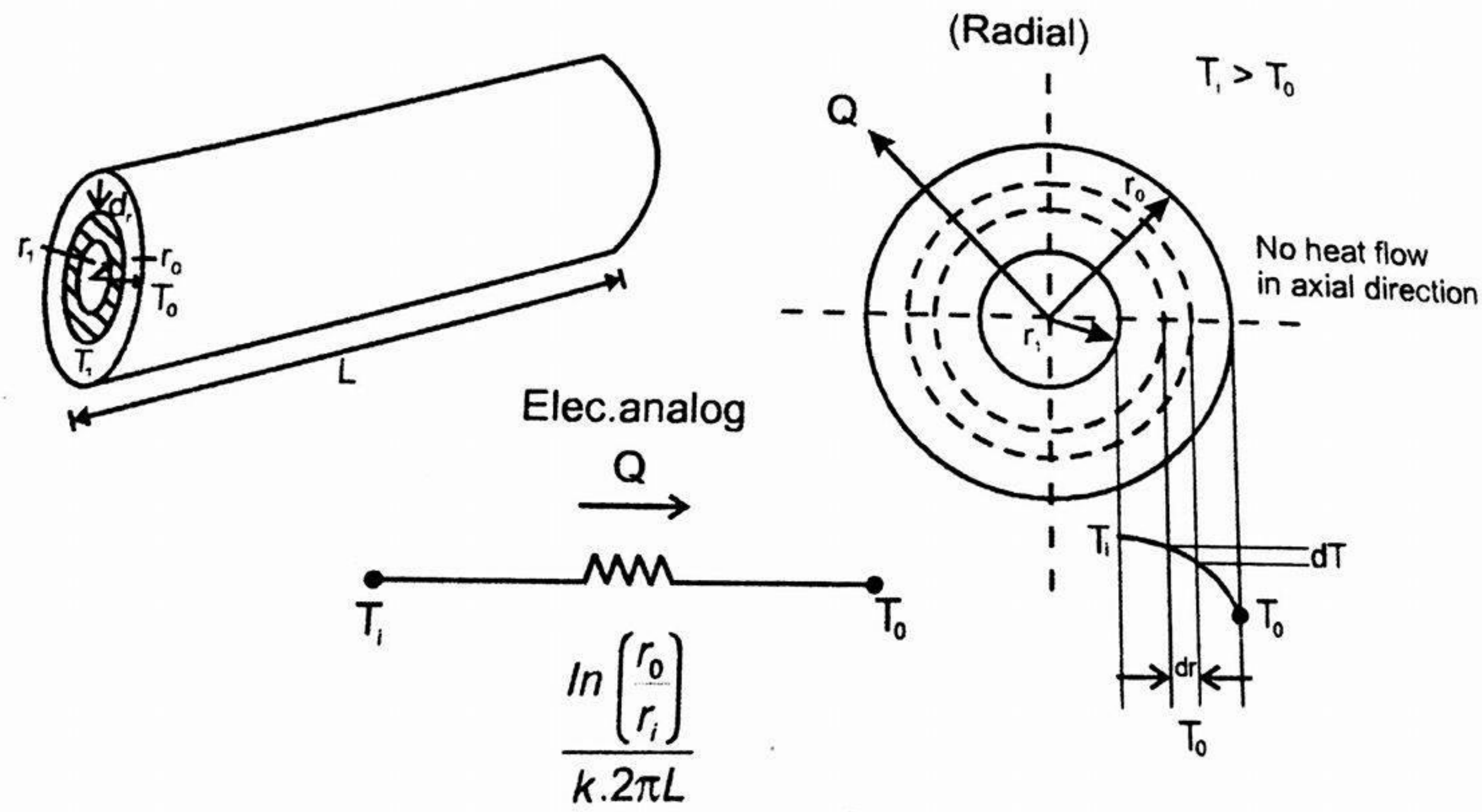
Since volume=v=dr.rdϕ.dz, hence by dividing both sides of the above equation by “k.V”, it is obtained that;

$$\frac{1}{r}.\frac{\partial}{\partial r}[\frac{r\partial T}{\partial r}]+\frac{1}{r^2}.\frac{\partial}{\partial \phi}[\frac{\partial T}{\partial \phi}]+\frac{\partial}{\partial z}[\frac{\partial T}{\partial z}]+\frac{q'''}{k}=\frac{1}{\alpha}.\frac{\partial T}{\partial \tau}\rightarrow(4)$$

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}.\frac{\partial T}{\partial r}\right]+\left[\frac{1}{r^2}.\frac{\partial^2 T}{\partial \phi^2}\right]+\left[\frac{\partial^2 T}{\partial z^2}\right]+\frac{q'''}{K}=\frac{1}{\alpha}.\frac{\partial T}{\partial \tau}\rightarrow(5)$$

Radial Heat Conduction Through a Cylinder without Internal Heat Generation

Consider a long hollow cylinder of length "L", inside radius r_i and outside radius r_o, having uniform thermal conductivity (k).



Radial Heat Conduction Through a Cylinder without Internal Heat Generation

Assumptions:

1. There is no internal generation in the cylinder.
2. The inside and outside temperatures T_i and T_o are constant.
3. It's a long cylinder. So the end losses are negligible.

From general heat conduction equation;

$$\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}.\frac{\partial T}{\partial r}\right] + \left[\frac{1}{r^2}.\frac{\partial^2 T}{d\phi^2}\right] + \left[\frac{\partial^2 T}{dz^2}\right] + \frac{q'''}{K} = \frac{1}{\alpha}.\frac{\partial T}{\partial \tau} \rightarrow (6)$$

Since there is steady state conduction in radial direction only without internal heat generation, all the other terms will become equal to zero. Hence equation (6) will become;

$$\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}.\frac{\partial T}{\partial r}\right] = 0 \Rightarrow \left[\frac{d^2 T}{dr^2} + \frac{1}{r}.\frac{dT}{dr}\right] \Rightarrow \frac{1}{r}.\frac{d}{dr}\left[r.\frac{dT}{dr}\right] = 0 \rightarrow (7)$$

Heat Conduction Through a Cylinder without Internal Heat Generation

Since $$\frac{1}{r} \neq 0 \Rightarrow \frac{d}{dr}\left[r.\frac{dT}{dr}\right] = 0 \rightarrow (8)$$

By double integrating equation (8) w.r.t. r;

$$\left[r.\frac{dT}{dr}\right] = C_1 \Rightarrow dT = C_1 r dr \Rightarrow T = C_1 \ln(r) + C_2 \rightarrow (9)$$

To find out C1 and C2, apply boundary conditions;

At $r = r_i$ $T = T_i$

At $r = r_o$ $T = T_o$

By applying boundary condition (1) and (2) in equation (9), we have;

$$T_i = C_1 \ln(r_i) + C_2 \rightarrow (10)$$
$$T_o = C_1 \ln(r_o) + C_2 \rightarrow (11)$$

Heat Conduction Through a Cylinder without Internal Heat Generation

By solving equation (10) and (11), we have;

$$C_1 = \frac{(T_i - T_o)}{\ln\left(\frac{r_i}{r_o}\right)} = \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)}$$

$$C_2 = T_i - \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)} \ln(r_i)$$

Heat Conduction Through a Cylinder without Internal Heat Generation

Substituting C_1 and C_2 in equation (9)

$$T = \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)} \ln(r) + T_i - \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)} \ln(r_i)$$

$$T = \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)} \ln(r) + \frac{T_i \ln r_o - T_i \ln r_i - T_o \ln r_i + T_i \ln r_i}{\ln\left(\frac{r_o}{r_i}\right)}$$

$$T = \frac{(T_o - T_i)}{\ln\left(\frac{r_o}{r_i}\right)} \ln(r) + \frac{T_i \ln r_o - T_o \ln r_i}{\ln\left(\frac{r_o}{r_i}\right)} \rightarrow (12)$$

This is called temperature distribution equation for cylinder system.

Heat Conduction Through a Cylinder without Internal Heat Generation

The heat transfer rate in the cylinder can be determined by integrating the Fourier law of heat conduction as;

$$Q = -kA\frac{dT}{dr} \Rightarrow Qdr = -kAdT \Rightarrow Qdr = -k(2\pi.r.L)dT$$

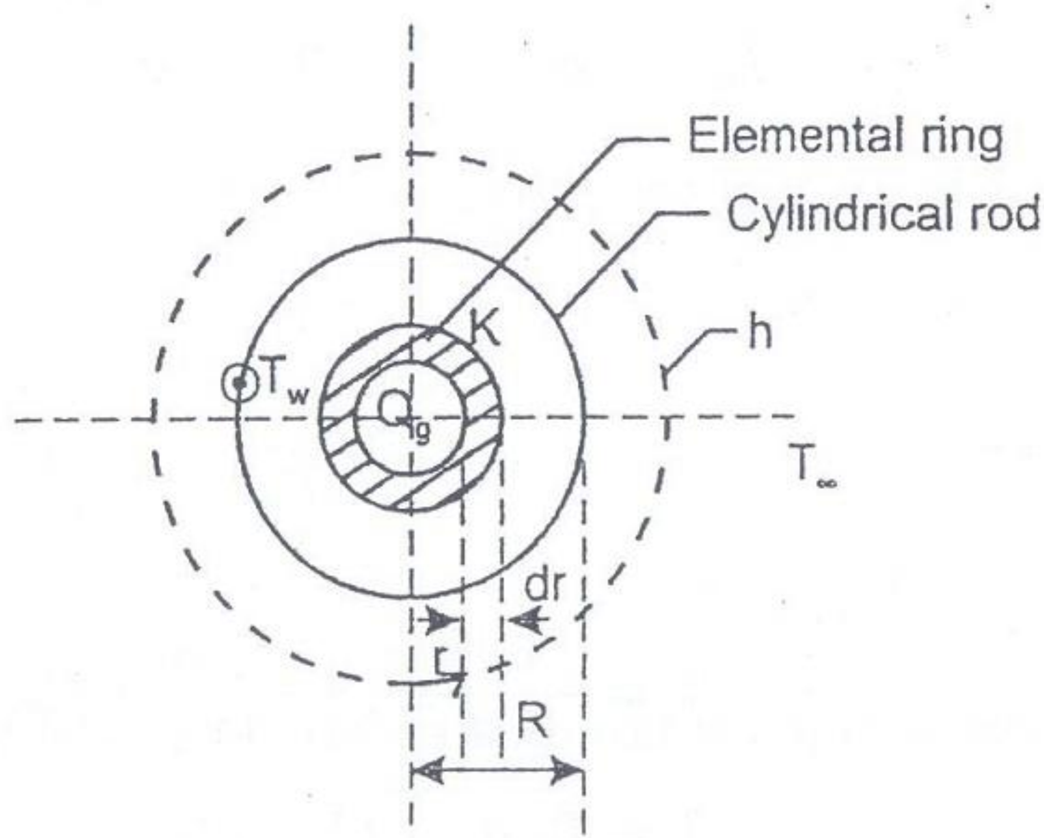
$$\Rightarrow Q\int_{r_i}^{r_o}\frac{dr}{r} = -2\pi.k.L\int_{T_i}^{T_o}dT \Rightarrow Q\ln\left(\frac{r_o}{r_i}\right) = -2\pi kL(T_o - T_i)$$

$$\Rightarrow Q = \frac{2\pi kL(T_i - T_o)}{\ln\left(\frac{r_o}{r_i}\right)} = \frac{(T_i - T_o)}{\ln\left(\frac{r_o}{r_i}\right)\Big/2\pi kL} = \frac{(T_i - T_o)}{R_{th}} \rightarrow (13)$$

Where R_{th} is the thermal resistance for hollow cylinder.

Radial Heat Conduction Through a Cylinder with Internal Heat Generation

Consider a cylinder of radius "R", having uniform thermal conductivity (k). Let an electric current passes through the cylinder, which causes uniform heat generation within the cylinder (q''' W/m^3).ing of radius "r" thickness "dr" and heat generated in the elemental ring as Qg.



Radial Heat Conduction Through a Cylinder with Internal Heat Generation

Assumptions:

1. There is internal generation in the cylinder.
2. The surface temperature and surrounding temperature are Tw and Tα respectively.
3. It's a long cylinder. So the end losses are negligible.

From general heat conduction equation;

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}.\frac{\partial T}{\partial r}\right]+\left[\frac{1}{r^2}.\frac{\partial^2 T}{d\phi^2}\right]+\left[\frac{\partial^2 T}{dz^2}\right]+\frac{q'''}{k}=\frac{1}{\alpha}.\frac{\partial T}{\partial \tau}\rightarrow(14)$$

Since there is steady state conduction in radial direction only with internal heat generation, all the other terms will become equal to zero. Hence equation (14) will become;

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}.\frac{\partial T}{\partial r}\right]+\frac{q'''}{k}=0\rightarrow(15)\Rightarrow\left[\frac{d^2 T}{dr^2}+\frac{1}{r}.\frac{dT}{dr}\right]+\frac{q'''}{k}=0$$

$$\Rightarrow\left[r\frac{d^2 T}{dr^2}+\frac{dT}{dr}\right]+\frac{q'''}{k}r=0\Rightarrow\frac{dT}{dr}\left[r.\frac{dT}{dr}\right]+\frac{q'''}{k}r=0\rightarrow(15)$$

Heat Conduction Through a Cylinder with Internal Heat Generation

By double integrating equation (15), we have;

$$\left[r.\frac{dT}{dr}\right]+\frac{q'''}{k}\frac{r^2}{2}=C_1 \Rightarrow \left[r.\frac{dT}{dr}\right]=-\frac{q'''}{k}\frac{r^2}{2}+C_1 \Rightarrow \left[\frac{dT}{dr}\right]=-\frac{q'''}{k}\frac{r}{2}+\frac{C_1}{r} \rightarrow (16)$$

$$T=-\frac{q'''}{k}\frac{r^2}{4}+C_1 \ln r + C_2 \rightarrow (17)$$

To find out C1 and C2, apply boundary conditions;

At $r = 0$ $dT/dr = 0$

At $r = R$ $T = T_w$

By applying boundary condition (1) in equation (16), we have;

$$C_1 = 0$$

By applying boundary condition (2) in equation (17), we have;

$$T_w=-\frac{q'''}{k}\frac{R^2}{4}+(0)\ln r + C_2 \Rightarrow T_w=-\frac{q'''}{k}\frac{R^2}{4}+C_2 \Rightarrow C_2 = T_w + \frac{q'''}{k}\frac{R^2}{4}$$

Heat Conduction Through a Cylinder with Internal Heat Generation

By putting values of C1 and C2 in equation (17)

$$T = -\frac{q'''}{k}\frac{r^2}{4} + (0)\ln r + T_w + \frac{q'''R^2}{4k} \Rightarrow T = T_w + \frac{q'''}{k}\left(R^2 - r^2\right) \rightarrow (18)$$

At center r=0, T = T_{max}

$$T_{max} = T_w + \frac{q'''R^2}{k} \rightarrow (18)$$

Surface Temperature: Heat Generated inside the cylinder

$$q''' = \frac{Q}{A} \Rightarrow Q = q'''A = q'''(\pi R^2 L) \rightarrow (19)$$

Heat due to convection on the surface of cylinder

$$Q = hA(T_w - T_\infty) = h(2\pi RL)(T_w - T_\infty) \rightarrow (20)$$

Heat Conduction Through a Cylinder with Internal Heat Generation

Equating equation 19 and 20,

$$\pi R^2 L q''' = h(2\pi RL)(T_w - T_\infty) \Rightarrow Rq''' = 2h(T_w - T_\infty) \Rightarrow 2hT_w = Rq''' + 2hT_\infty$$

$$T_w = T_\infty + \frac{Rq'''}{2h} \rightarrow (21)$$

Conduction Heat Transfer-Class Problems

Example 3.1:

A hollow cylinder 5 cm I.D and 10 cm O.D has an inner surface, temperature of 200^0C and an outer surface temperature of 100^0C. Determine the temperature of the point half way between the inner and outer surfaces. If the thermal conductivity of cylinder material is $70W/mK$, determine also the heat flow through the cylinder per linear meter.

Conduction Heat Transfer-Class Problems

Solution

$$Q = \frac{-kA(T_0 - T_i)}{\ln\left(\frac{r_0}{r_i}\right)}$$

$$= \frac{kA(T_i - T_0)}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{k \cdot 2\pi L(T_i - T_0)}{\ln\left(\frac{r_0}{r_i}\right)}$$

$$\frac{Q}{L} = \frac{70 \times 2 \times \pi \times (200 - 100)}{\ln\left(\frac{5}{2.5}\right)}$$

$$\boxed{\frac{Q}{L} = 63434.3\ W/m} = 63.43\ kW/m$$

Conduction Heat Transfer-Class Problems

At half-wave between $r_i \text{and} r_0 \therefore r' = \frac{r_i + r_0}{2} = \frac{5 + 2.5}{2} = 3.75cm$

$$Q = \frac{2\pi kL(T_i - T_0)}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{2\pi kL(T_i - T)}{\ln\left(\frac{r'}{r_i}\right)}$$

$$(T_i - T) = \frac{(T_i - T_0)}{\ln\left(\frac{r'}{r_i}\right)} = \frac{2\pi kL(T_i - T)}{\ln\left(\frac{r^1}{r_i}\right)}$$

$$(T_i - T) = \frac{(T_1 - T_0)\ln\left(\frac{r'}{r_i}\right)}{\ln\left(\frac{r_0}{r_i}\right)}$$

$$= \frac{(200 - 100)\ln\left(\frac{3.75}{2.50}\right)}{\ln\left(\frac{5.00}{2.50}\right)} = 57.72$$

$$\therefore \quad T = T_i - 57.72 = 200 - 57.72$$

$$\boxed{T = 142.28^\circ C}$$

Conduction Heat Transfer-Class Problems

Example 3.2

A copper wire of 40 mm diameter carries 250 A and has a resistance of $0.25 \times 10^{-4} \Omega$cm/length surface temperature of copper wire is 250°C, and the ambient air temperature is 10°C. If the thermal conductivity of the copper wire is 175 W/mK. Calculate

1. Heat transfer co-efficient
2. Maximum temperature

Conduction Heat Transfer-Class Problems

Given:

$$D = 0.040\, m$$
$$R = 0.020\, m$$
$$I = 250\ A$$
$$R_t = 0.25 \times 10^{-4} \Omega\ cm/length$$
$$T_w = 250 + 273 = 523\ K$$
$$T_\infty = 10 + 273 = 283\ K$$
$$k = 175\ W/mK$$

Find:

(1) h (2) $T_{\max}$

Conduction Heat Transfer-Class Problems

Solution

We know that

$$\text{Heat transfer } Q = I^2 R_t$$
$$= (250)^2 \times (0.25 \times 10^{-4})$$
$$= 1.562\ W/cm$$
$$Q = 1.562 \times 10^2\ W/m$$
$$\boxed{Q = 156\ W/m}$$

We know that

$$\text{Heat generated } q''' = \frac{Q}{V} = \frac{156}{\pi R^2 L}$$
$$= \frac{156}{\pi (0.020)^2}$$

$$\boxed{q''' = 124140\ W/m^3}$$

Conduction Heat Transfer-Class Problems

∴ **Maximum Temperature**

$$T_{max} = T_w + \frac{q''' R^2}{4k}$$

$$= 523 + \frac{124140 \times (0.020)^2}{4 \times 175}$$

$$\boxed{T_{max} = 523.07\ K}$$

To find heat transfer coefficient

$$T_w = T_\infty + \frac{q''' R}{2h}$$

$$523 = 283 + \frac{0.020 \times 124140}{2 \times h}$$

$$\boxed{h = 5.17\ W/m^{2\circ}C}$$

Links for Video Lectures

Theory Class:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20THEORY-20210426_093607-Meeting%20Recording.mp4?web=1

Lab Session_03:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20LAB-20210426_120950-Meeting%20Recording.mp4?web=1


Thank you!