Heat & Mass Flow Processes

Week_02

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Basic Equations to Conduction

Introduction:

- Conduction is a mode of heat transfer that occurs in solid and fluids when there is no bulk movement of fluid.
- The basic equation for conduction is given by Fourier which states that the conduction heat transfer in a solid in a particular direction is directly proportional to the area normal to heat transfer and the temperature gradient in that direction.
- From Fourier law, it is clear that heat transfer has magnitude and direction.

Basic Equations to Conduction

Heat diffusion Equation In Cartesian Coordinate system

(X,Y,Z Coordinates)

- Let us consider a small element of a cube of sides dx, dy, dz as shown in Fig. 1 in which heat enters from three faces and leaves the object from the other three faces in x, y and z directions respectively.

Heat diffusion Equation In Cartesian Coordinate system

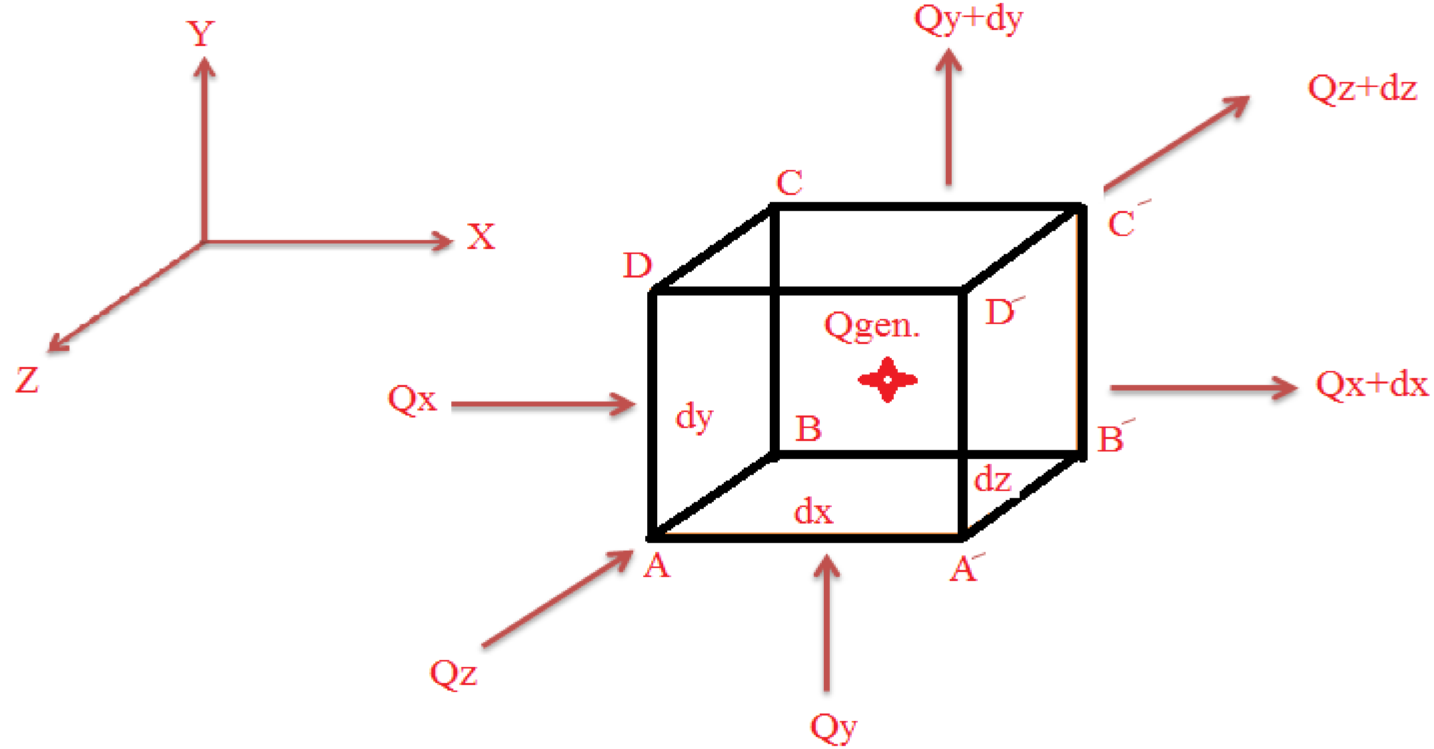


Fig. 1: Elemental volume for the dimensional heat-conduction analysis in Cartesian coordinates

Coordinates:	X	:	Y	:	Z
Heat Transfer:	Qx	:	Qy	:	Qz
Heat Flux (q=Q/A)	qx	:	qy	:	qz
Thickness:	dx	:	dy	:	dz

Heat diffusion Equation In Cartesian Coordinate system

- Energy balance for the small element is obtained from the First law of Thermodynamics.

(Net heat conducted into the element in dx dy dz per unit time I) + (Internal heat generated per unit time II)

= (Increase in internal energy per unit time III) + (work done by element per unit time IV)

- The IV term is small because the work done by the element due to change in temperature in neglected.

Qin– Qout + Qgen. = Rate of change in Internal Energy

$$(Q_x + Q_y + Q_z) - (Q_{x+dx} + Q_{y+dy} + Q_{z+dz}) + (Qgen.) = mc\frac{\partial T}{\partial \tau}$$

$$(Q_x + Q_y + Q_z) - (Q_{x+dx} + Q_{y+dy} + Q_{z+dz}) + (Qgen.) = \rho vc\frac{\partial T}{\partial \tau} \rightarrow (1)$$

Heat diffusion Equation In Cartesian Coordinate system

Let

- q_x be the heat flux in "x" direction at face ABCD and q_{x+dx} be the heat flux at x+dx direction at face A'B'C'D'.
- q_y be the heat flux in "y" direction at face ABB'A' and q_{y+dy} be the heat flux at y+dy direction at face DCC'D'.
- q_z be the heat flux in "z" direction at face ADD'A' and q_{z+dz} be the heat flux at z+dz direction at face BCC'B'.

Now the rate of heat transfer in 'x', 'y', and 'z' direction is given by:

$$\left.\begin{aligned} Q_x &= q_x d_y d_z = -k_x \frac{\partial T}{\partial x} d_y d_z \\ Q_y &= q_y d_x d_z = -k_y \frac{\partial T}{\partial y} d_x d_z \\ Q_z &= q_z d_x d_y = -k_z \frac{\partial T}{\partial z} d_x d_y \end{aligned}\right\} \rightarrow (2)$$

Heat diffusion Equation In Cartesian Coordinate system

Similarly, the rate of heat transfer in ‘x+dx’, ‘y+dy’, and ‘z+dz’ direction is given by:

$$\left.\begin{aligned} Q_{x+dx} &= Q_x + \frac{\partial Q}{\partial x} d_x \ldots\ldots \\ Q_{y+dy} &= Q_y + \frac{\partial Q}{\partial y} d_y \ldots\ldots \\ Q_{z+dz} &= Q_z + \frac{\partial Q}{\partial z} d_z \ldots\ldots \end{aligned}\right\} \rightarrow (3)$$

Now by putting Equation (3) in Equation (1) we have.

$$(Q_x + Q_y + Q_z) - (Q_x + \frac{\partial Q}{\partial x} d_x + Q_y + \frac{\partial Q}{\partial y} d_y + Q_z + \frac{\partial Q}{\partial z} d_z) + (Qgen.) = \rho vc \frac{\partial T}{\partial \tau}$$

$$-(\frac{\partial Q}{\partial x} d_x) - (\frac{\partial Q}{\partial y} d_y) - (\frac{\partial Q}{\partial z} d_z) + (Qgen.) = \rho vc \frac{\partial T}{\partial \tau} \rightarrow (4)$$

Heat diffusion Equation In Cartesian Coordinate system

Now by putting values of Qx, Qy, and Qz in Equation (4) we have.

$$-\frac{\partial}{\partial x}\left[-k_x\frac{\partial T}{\partial x}d_y d_z\right]d_x-\frac{\partial}{\partial y}\left[-k_y\frac{\partial T}{\partial y}d_x d_z\right]d_y-\frac{\partial}{\partial z}\left[-k_z\frac{\partial T}{\partial z}d_x d_y\right]d_z+\left[q'''.V\right]=\rho vc\frac{\partial T}{\partial \tau}$$

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial T}{\partial x}d_x d_y d_z\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial T}{\partial y}d_x d_y d_z\right]+\frac{\partial}{\partial z}\left[k_z\frac{\partial T}{\partial z}d_x d_y d_z\right]+\left[q'''.V\right]=\rho vc\frac{\partial T}{\partial \tau}$$

$$k_x d_x d_y d_z\frac{\partial}{\partial x}\left[\frac{\partial T}{\partial x}\right]+k_y d_x d_y d_z\frac{\partial}{\partial y}\left[\frac{\partial T}{\partial y}\right]+k_z d_x d_y d_z\frac{\partial}{\partial z}\left[\frac{\partial T}{\partial z}\right]+\left[q'''.V\right]=\rho vc\frac{\partial T}{\partial \tau}$$

Since $k_x=k_y=k_z=k$ and $d_x d_y d_z=V$, Hence by dividing equation (5) by k and V;

$$\frac{\partial}{\partial x}\left[\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial z}\left[\frac{\partial T}{\partial z}\right]+\left[\frac{q'''}{k}\right]=\frac{\rho c}{k}\frac{\partial T}{\partial \tau}\rightarrow(5)$$

Equation (5) is known as “General Heat conduction Equation in Cartesian Coordinate System”. Since k/ρc is thermal diffusivity and is denoted by α, hence the above equation becomes;

Heat diffusion Equation In Cartesian Coordinate system

$$\frac{\partial}{\partial x}\left[\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial z}\left[\frac{\partial T}{\partial z}\right]+\left[\frac{q'''}{k}\right]=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}\rightarrow (6)$$

If the element is very small then these partial derivatives in equation (6) can be written as;

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q'''}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}\rightarrow (7)$$

“Thermal diffusivity tells us how fast heat is propagated”. If it is large then, heat flow takes place quickly with less time.

$$\alpha=\frac{ThermalConductivity}{ThermalCapacity}=\frac{k}{\rho c}$$

Heat conduction through a plane wall without internal heat generation (SLAB)

Consider a slab of thickness L in x direction, having uniform thermal conductivity (k) as shown in Fig. 2.

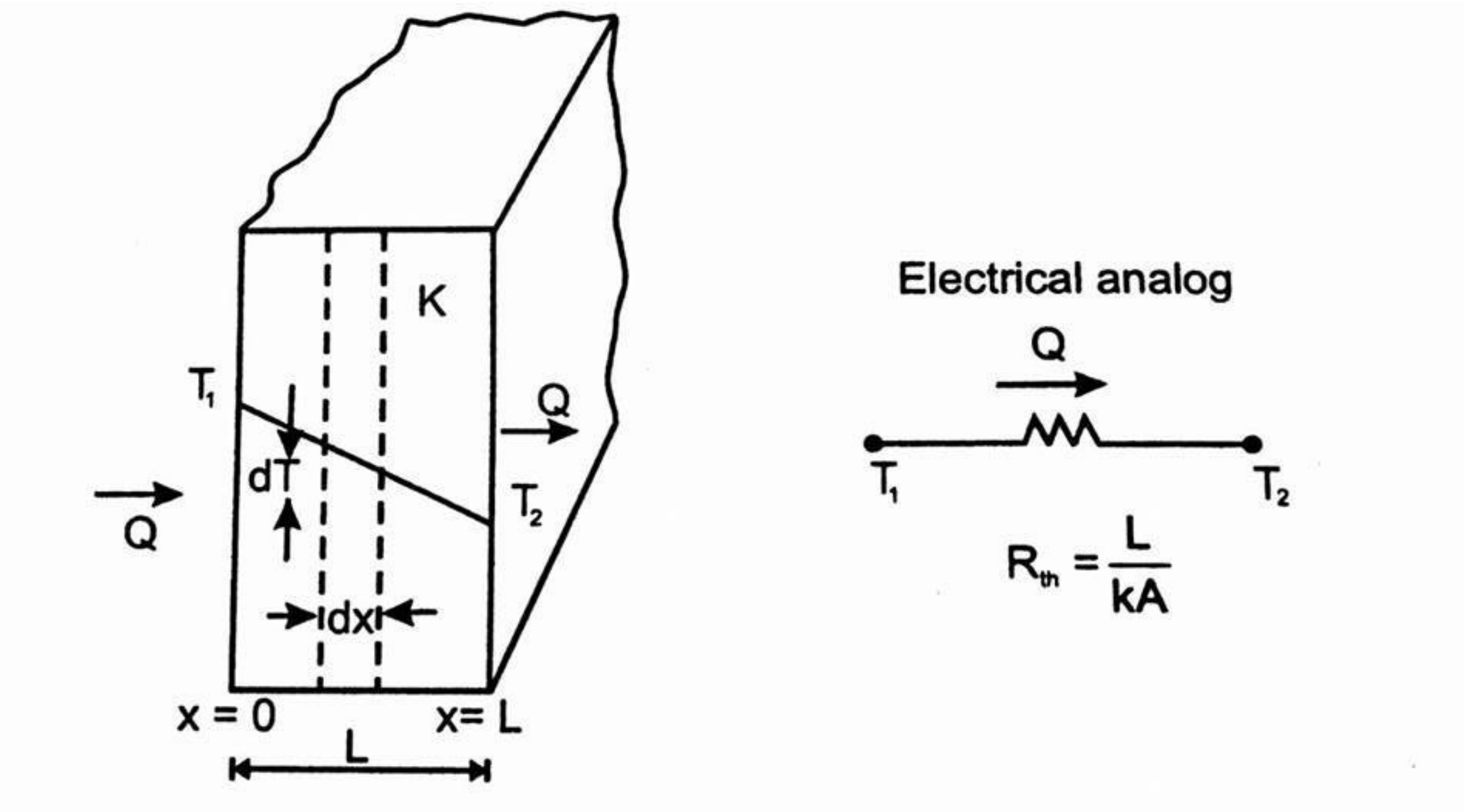


Fig. 2: One dimensional heat conduction through a plane wall without internal generation

Heat conduction through a plane wall without internal heat generation (SLAB)

Assumptions:

1. There is no internal generation in the slab and the sides are at constant temperature T_1 and T_2 at x = 0 x = L respectively.
2. One dimensional heat conduction, hence temperature is a function of x only.
3. Energy loss through the edges are negligible.

From general heat conduction equation (eq. 7), we have;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \rightarrow (8)$$

Since there is steady state conduction in x direction only, all the other terms will become equal to zero. Hence equation (8) will become;

Heat conduction through a plane wall without internal heat generation (SLAB)

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{d^2 T}{dx^2} = 0 \rightarrow (9)$$

By double integrating equation (9) w.r.t. x;

$$\frac{dT}{dx} = C_1 \Rightarrow T = C_1 x + C_2 \rightarrow (10)$$

The boundary conditions are;

$$T = T_1 \quad \text{at} \quad x = 0$$

$$T = T_2 \quad \text{at} \quad x = L$$

By applying boundary condition 1 in equation (10) we have;

$$T_1 = C_2$$

By applying boundary condition 2 in equation (10) we have;

$$T_2 = C_1 L + T_1 \Rightarrow C_1 = \frac{T_2 - T_1}{L}$$

Substituting the values of C_1 and C_2 in equation (10), we have;

$$T = \frac{T_2 - T_1}{L} x + T_1 \rightarrow (11)$$

Heat conduction through a plane wall without internal heat generation (SLAB)

The above equation is called "temperature equation". The heat transfer rate in the slab can be determined from the Fourier law of heat conduction as;

$$Q = -kA\frac{dT}{dx} = \frac{kA}{L}(T_1 - T_2) = \frac{(T_1 - T_2)}{L/kA} = -\frac{dT}{R_{cond}} \rightarrow (12)$$

Equation (12) is known as general heat conduction equation in a plane wall without internal generation.

Heat conduction through a plane wall with internal heat generation (SLAB)

Consider a slab of thickness "L" in "x" direction, having uniform thermal conductivity (k) as shown in Fig. 3.

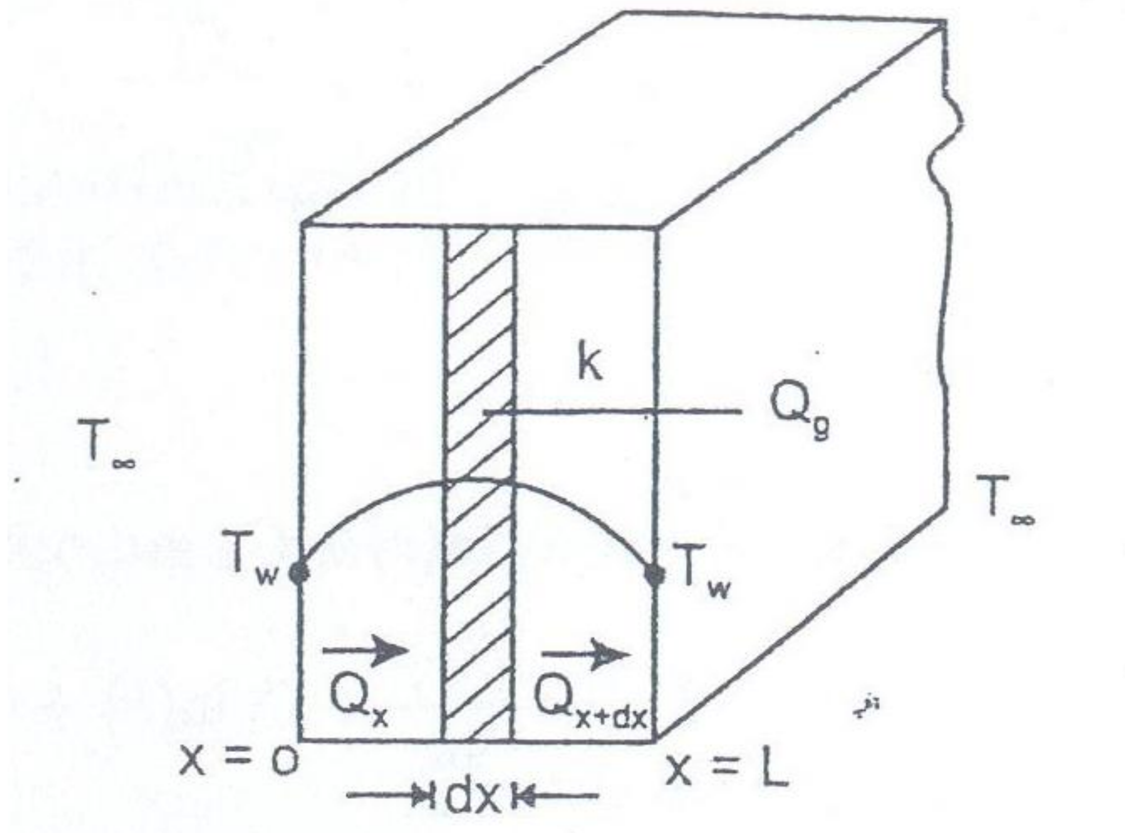


Fig. 3: One dimensional heat conduction through a plane wall with internal generation

Heat conduction through a plane wall with internal heat generation (SLAB)

From general heat conduction equation (eq. 7), we have;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \rightarrow (13)$$

Assumptions:

1. There is an internal generation in the slab and the sides are at constant temperature T_w at x = 0 and x = L respectively.
2. One dimensional heat conduction, hence temperature is a function of "x" only.
3. Energy loss through the edges are negligible.

Since there is steady state conduction in "x" direction only with internal generation, all the other terms will become equal to zero. Hence equation (13) will become;

Heat conduction through a plane wall internal heat generation (SLAB)

$$\frac{\partial^2 T}{\partial x^2}+\frac{q'''}{k}=0 \Rightarrow \frac{d^2T}{dx^2}+\frac{q'''}{k}=0 \rightarrow (14)$$

By double integrating equation (14) w.r.t. x;

$$\frac{dT}{dx}=-\frac{q'''}{k}x+C_1 \Rightarrow T=-\frac{1}{2}\frac{q'''}{k}x^2+C_1x+C_2 \rightarrow (15)$$

The boundary conditions are;

$T = T_w$ at $x = 0$ and $x = L$

By applying boundary condition 1 in equation (15) we have;

$$T_w = C_2$$

By applying boundary condition 2 in equation (15) we have;

$$C_1=\frac{1}{2}\frac{q'''}{k}L$$

Substituting the values of C_1 and C_2 in equation (15), we have;

$$T=-\frac{1}{2}\frac{q'''}{k}x^2+\frac{1}{2}\frac{q'''}{k}Lx+T_w \Rightarrow T=\frac{1}{2}\frac{q'''}{k}(L-x)x+T_w \rightarrow (16)$$

Heat conduction through a plane wall with internal heat generation (SLAB)

To obtain the maximum temperature which is at the center by putting x = L/2 in equation(16);

$$T_{\max} = T_w + \frac{1}{8}\frac{q'''L^2}{k} \rightarrow (17)$$

To find out the wall temperature, we know that the heat flow rate for "first half of the wall" and for "second half of the wall";

$$Q = \frac{q'''}{2}AL$$

We know that the heat transfer by convection on the two faces;

$$Q = hA(T_w - T_\infty) \Rightarrow \frac{q'''}{2}AL = hA(T_w - T_\infty) \Rightarrow hATw = hAT\infty + \frac{q'''}{2}AL$$

$$Tw = T\infty + \frac{q'''L}{2h} \rightarrow (18)$$

T_w = Surface or wall temperature.

Conduction Heat Transfer-Class Problems

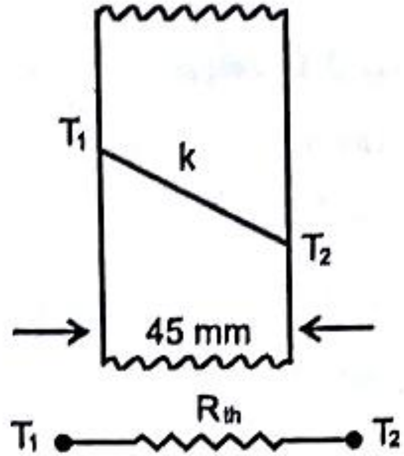
Example 2.1:

Calculate the rate of heat transfer per unit area through a copper plate of 45mm thick, whose one face is maintained at 350°C and other face at 50°C. Take $k = 370\ W/mK$

Conduction Heat Transfer-Class Problems

Given:

Copper plate



$$L = 45mm = 0.045m$$
$$T_1 = 350°C$$
$$T_2 = 50°C$$
$$k = 370W/mK$$

Find:

Rate of heat transfer per unit area $\frac{Q}{A} = q$

Conduction Heat Transfer-Class Problems

Solution

We know that

$$Q = -kA\frac{dT}{dx}$$

$$\frac{Q}{A} = \frac{-k(T_2 - T_1)}{L}$$

$$= \frac{-370(50 - 350)}{0.045}$$

$$\boxed{\frac{Q}{A} = q = 2.467 \times 10^6\ W/m^2}$$

Conduction Heat Transfer-Class Problems

Example 2.2:

A plane wall (thermal conductivity $= 10.2 W/mK$) of 100 mm thickness and area $3m^2$ has steady surface temperature of $170°C$. The temperature of other side is $100°C$.

Determine

1. The rate of heat flow across the plane wall.
2. The temperature gradient in the flow direction.

Conduction Heat Transfer-Class Problems

Given

ane wall

$$k = 10.2\,W/mK$$
$$L = 100\,mm = 100 \times 10^{-3}\,m^2$$
$$A = 3\,m^2$$
$$T_1 = 170^0\,C$$
$$T_2 = 100^0\,C$$

Find

$a)\ Q$ and $b)\ \dfrac{dT}{dx}$

Conduction Heat Transfer-Class Problems

Solution

We know that

1.

$$Q = -kA \cdot \frac{dT}{L}$$

$$Q = -10.2 \times 3 \left[\frac{100 - 170}{0.1}\right]$$

$$\boxed{Q = 21.42\, kW}$$

In order to find Temperature gradient.

2.

$$\frac{dT}{dx} = \frac{Q}{kA} = \frac{21.42 \times 1000}{10.2 \times 3} = -700K/m$$

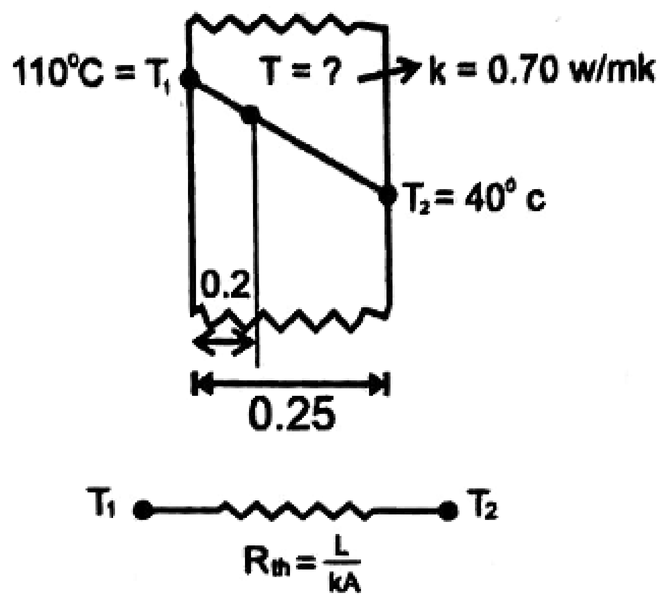
$$\therefore \quad \boxed{\frac{dt}{dx} = -700K/m}$$

Conduction Heat Transfer-Class Problems

Example 2.3:

Calculate the rate of heat loss from a red brick wall of length 5 m, height 4m and thickness 0.25m. The temperature of the inner surface is 110° C and that of outer surface is 40°C. The thermal conductivity of red brick $k = 0.70W/mK$.

Calculate also the temperature at an interior point of the wall 20 cm. distance from the inner wall.



Conduction Heat Transfer-Class Problems

Solution

a) $$Q = -kA \cdot \frac{dT}{dx}$$

$$= \frac{-(0.70)(5 \times 4)(40 - 110)}{0.25}$$

$$\boxed{Q = 3920\, W}$$

At $x = 0.20\, m$. Temperature at an interior point is

b) $$T = \frac{(T_2 - T_1)}{L} \cdot x + T_1$$

$$= \frac{(40 - 110) \times 0.20}{0.25} + 110$$

$$= -56 + 110$$

$$\boxed{T = 54°C}$$

Conduction Heat Transfer-Class Problems

Example 2.4:

A plane wall 10 cm thick generates heat at the rate of 4×10^4 W/m^3 when an electric current is passed through it. The convective heat transfer coefficient between each face of the wall and the ambient air is 50 W/m^{2}K. Determine

a. the surface temperature

b. the maximum air temperature on the wall. Assume the ambient air temperature to be 20°C and the thermal conductivity of the wall material to be 15 W/mK.

Conduction Heat Transfer-Class Problems

Given:

$$\text{Thickness } L = 0.10\ m$$

$$\text{Heat generated } q''' = 4 \times 10^4\ W/m^3$$

$$h = 50\ W/m^2K$$

$$T_\infty = 20 + 273 = 293\ K$$

$$k = 15\ W/mK$$

Find:

(1) T_w (2) $T_{\max}$

<u>*Conduction Heat Transfer-Class Problems*</u>

Solution

Heat conducted with internal heat generation

$$\text{Surface temperature } T_w = T_\infty + \frac{q''' L}{2h}$$

$$= 293 + \frac{4 \times 10^4 \times 0.10}{2 \times 50}$$

$$\boxed{T_w = 333\ K}$$

$$\text{Maximum temperature } T_{\max} = T_w + \frac{q''' L^2}{8k}$$

$$= 333 + \frac{4 \times 10^4 \times (0.10)^2}{8 \times 15}$$

$$\boxed{T_{\max} = 336.3\ K}$$

Conduction Heat Transfer-Class Problems

Example 2.5:

A concrete wall of 1m thick is poured with concrete. The hydration of concrete generates 150 W/m^3 heat. If both the surfaces are maintained at 35°C find the maximum temperature in the wall.

Given:

$$L = 1\ m$$

$$q = 150\ W/m^3$$

$$T_w = 35 + 373 = 308\ K$$

Find:

$T_{\max}$

Conduction Heat Transfer-Class Problems

Solution

$$T_{\max} = T_w + \frac{qL^2}{8k}$$

Thermal conductivity $k = 1279 \times 10^{-3}$ W/mK
(from HMT Data book,)

$$\therefore \quad T_{\max} = 308 + \frac{150(1)^2}{8 \times 1279 \times 10^{-3}}$$

$$\boxed{T_{\max} = 322.6\ K}$$

Links for Video Lectures

Theory Class:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20THEORY-20210419_094847-Meeting%20Recording.mp4?web=1

Lab Session_02:

https://pern.sharepoint.com/sites/HMFP-MECH-TECH-2018/Shared%20Documents/General/Recordings/HMFP%20LAB-20210419_140018-Meeting%20Recording.mp4?web=1


Thank You!